ON STEINER SETS AND STEINER POLYNOMIALS OF SQUARE OF PATHS

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ABSTRACT

In this paper, we study the concept of Steiner domination polynomial of square of path P_n^2. The Steiner domination polynomial of P_n^2 is the polynomial $S(P_n^2, x) = \sum_{i=s(P_n^2)}^{n} s(P_n^2, i)x^i$, where $s(P_n^2, i)$ is the number of Steiner dominating sets of P_n^2 of size i and $s(P_n^2)$ is the Steiner domination number of P_n^2. We obtain some properties of $S(P_n^2, x)$ and its coefficients.

Key words: Steiner set, Steiner number, Steiner polynomial.

I. Introduction

For a connected graph G and a set $W \subseteq V(G)$, a tree contained in G is a Steiner tree with respect to W if T is a tree of minimum order with $W \subseteq V(G)$. The set $S(W)$ contains, of all vertices in G that lie on some Steiner tree with respect to W. The minimum cardinality among the Steiner sets of G is the Steiner number, $s(G)$. We denote the family of Steiner sets of a connected graph G with cardinality i by $S(G, i)$.

Each extreme vertex of a graph G belongs to every Steiner set of G. In particular, each end-vertex of G belongs to every Steiner set of G. $[n]$ denotes the set of all positive integers less than or equal to n.

2. Steiner Sets and Polynomial of a Square of Path

Definition 2.1

Let P_n^2, $n \geq 2$, be a path of n vertices. Let $V(P_n^2) = \{1, 2, 3, \ldots, n\}$ and $E(P_n^2) = \{(1, 2), (2, 3), \ldots, (n-1, n), (1, 3), (2, 4), \ldots\ldots(n-3, n-1), (n-2, n)\}$. Let $S(P_n^2, i)$ be the family of Steiner sets of P_n^2 with cardinality i and let $s(P_n^2, i) = |S(P_n^2, i)|$. Then the Steiner polynomial, $S(P_n^2, x)$ of P_n^2 is defined as

$$S(P_n^2, x) = \sum_{i=s(P_n^2)}^{n} s(P_n^2, i)x^i \text{ , where } s(P_n^2) \text{ is the Steiner number of } P_n^2.$$

Lemma 2.2

For any square of path P_{2n}^2, $n \geq 1$, with $\left|V(P_{2n}^2)\right| = 2n$, the Steiner number, $s(P_{2n}^2) = 2$ and for any square of path P_{2n+1}^2, $n \geq 1$, with $\left|V(P_{2n+1}^2)\right| = 2n + 1$, the Steiner number, $s(P_{2n+1}^2) = 3$.

Proof:

For any square of path P_{2n}^2, $n \geq 1$, the minimum Steiner set is $\{1, 2n\}$. Therefore, $s(P_{2n}^2) = 2$.

Also, for any square of path P_{2n+1}^2, the minimum Steiner sets are $\{1, 2, 2n + 1\}$, $\{1, 4, 2n + 1\}$, $\{1, 6, 2n + 1\}$, …, $\{1, 2n, 2n + 1\}$. Therefore, $s(P_{2n+1}^2) = 3$. □

Lemma 2.3

Let P_n^2, $n \geq 3$, be a square of path of n vertices. Then:

(i) $s(P_n^2, i) = 0$ if $i < s(P_n^2)$ or $i > n$,

(ii) $s(P_n^2, i) = 0$ if $n + i$ is odd and

(iii) $s(P_n^2, i) > 0$ if $n + i$ is even and $s(P_n^2) \leq i \leq n$.

Proof:

(i) It follows from lemma 2.2 and the definition of Steiner set.

(ii) If either n is odd and i is even or n is even and i odd, then there is no set $W \subseteq V(P_n^2)$ such that trees T of minimum order with $W \subseteq V(T)$ containing all the vertices of P_n^2.

(iii) There are two cases.

If both n and i are odd, then $\{1, 2, 3, i - 1, n\}$ is a Steiner set. Hence, $s(P_n^2, i) > 0$.

If both n and i are even, then also $\{1, 2, 3, i - 1, n\}$ is a Steiner set. Hence, $s(P_n^2, i) > 0$. □

Lemma 2.4

Let P_n^2, $n \geq 3$, be a square of path of n vertices. Then:

(i) $S(P_n^2, i) = \phi$ if $i < s(P_n^2)$ or $i > n$

(ii) $S(P_n^2, x)$ has no constant term and first degree terms.

(iii) $S(P_n^2, x)$ is a strictly increasing function on $[0, \infty)$.

Proof is obvious.

Lemma 2.5

Let P_n^2, $n \geq 3$, be a square of path of n vertices.

(i) If $S(P_{n-1}^2, i-1) = \phi$ then $S(P_n^2, i) = \phi$.

(ii) If $S(P_{n-1}^2, i-1) \neq \phi$ then $S(P_n^2, i) \neq \phi$.

Proof:

(i) If $S(P_{n-1}^2, i-1) = \phi$ then $(n-1) + (i-1)$ is odd $\Rightarrow n + i$ is odd

Hence, $S(P_n^2, i) = \phi$.

(ii) If $S(P_{n-1}^2, i-1) \neq \phi$ then $(n-1) + (i-1)$ is even

$\Rightarrow n + i$ is even

Hence, $S(P_n^2, i) \neq \phi$. □

Theorem 2.6

Let P_n^2, $n \geq 4$, be a square of path of n vertices. Let $S(P_n^2, i)$ be the family of Steiner sets of cardinality i and $|S(P_n^2, i)| = s(P_n^2, i)$, then

$$s(P_n^2, i) = s(P_{n-1}^2, i-1) + s(P_{n-2}^2, i).$$

Proof:

There are two cases.

Case (i):

If $n + i$ is odd, then both $(n - 1) + (i - 1)$ and $(n - 2) + i$ are odd.

Hence, $s(P_n^2, i) = s(P_{n-1}^2, i-1) = s(P_{n-2}^2, i) = 0$, by part (ii) of lemma 2.3.

Therefore, $s(P_n^2, i) = 0 = s(P_n^2, i-1) + s(P_n^2, i)$

Case (ii):

Let $n + i$ be even. Then, by part (iii) of lemma 2.3, $s(P_{n-2}^2, i) > 0$.

Consider a member S_1 of $S(P_{n-2}^2, i)$. It is a Steiner set of P_{n-2}^2 with cardinality i. Every Steiner set contains all the extreme vertices of the graph. So, it contains the first vertex and the last vertex. All the $(n-2)$ vertices lie on some Steiner trees of the set S_1. We remove the $(n-2)^{th}$ vertex and add n^{th} vertex to S_1. This is a Steiner set and a member of $S(P_n^2, i)$, because all the n vertices of P_n^2 lie on some of its Steiner trees.

Now we take a member S_2 of $S(P_{n-1}^2, i-1)$. The set contains $(i-1)$ vertices and all the $(n-1)$ vertices lie on some of its Steiner trees starting from the first vertex to the last vertex $(n-1)$. Add the n^{th} vertex to this set. This set now has i vertices and it is a member of $S(P_n^2, i)$ since every Steiner tree of P_{n-1}^2 is a Steiner tree of P_n^2 by adding a vertex n to it.

Therefore, $S(P_{n-1}^2, i-1) \cup S(P_{n-2}^2, i) \subseteq S(P_n^2, i)$ ……………………………….. (1)

Consider a member of $S(P_n^2, i)$.

If it contains $(n-1)^{th}$ vertex, the removal of the n^{th} vertex from the set results a Steiner set of P_{n-1}^2 with cardinality i-1. If it contains neither $(n-1)^{th}$ vertex nor $(n-2)^{th}$ vertex removal of the n^{th} vertex and adding $(n-2)^{th}$ vertex results a Steiner set and a member of $S(P_{n-2}^2, i)$.

Hence, every Steiner set of P_n^2 of cardinality i is got either from the Steiner sets of P_{n-1}^2 of cardinality $(i-1)$ or from the Steiner sets of P_{n-2}^2 of cardinality i.

Therefore, $S(P_n^2, i) \subseteq S(P_{n-1}^2, i-1) \cup S(P_{n-2}^2, i)$ ….……………………….. (2)

From (1) and (2), we get $S(P_n^2, i) = S(P_{n-1}^2, i-1) \cup S(P_{n-2}^2, i)$

Hence, $s(P_n^2, i) = s(P_{n-1}^2, i-1) + s(P_{n-2}^2, i)$. □

Theorem 2.7

Let P_n^2, $n \geq 4$, be a square of path of n vertices. Then its Steiner polynomial is

$$S(P_n^2, x) = x\,S(P_{n-1}^2, x) + S(P_{n-2}^2, x),$$

with the initial values $S(P_2^2, x) = x^2$ and $S(P_3^2, x) = x^3$.

Proof:

By theorem 2.6, we have

$$s(P_n^2, i) = s(P_{n-1}^2, i-1) + s(P_{n-2}^2, i).$$

When $i = 3$,

$$s(P_n^2, 3) = s(P_{n-1}^2, 2) + s(P_{n-2}^2, 3)$$

$$\Rightarrow x^3 s(P_n^2, 3) = x^3 s(P_{n-1}^2, 2) + x^3 s(P_{n-2}^2, 3)$$

When $i = 4$,

$$s(P_n^2, 4) = s(P_{n-1}^2, 3) + s(P_{n-2}^2, 4)$$

$$\Rightarrow x^4 s(P_n^2, 4) = x^4 s(P_{n-1}^2, 3) + x^4 s(P_{n-2}^2, 4)$$

When $i = 5$,

$$s(P_n^2, 5) = s(P_{n-1}^2, 4) + s(P_{n-2}^2, 5)$$

$$\Rightarrow x^5 s(P_n^2, 5) = x^5 s(P_{n-1}^2, 4) + x^5 s(P_{n-2}^2, 5)$$

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When $i = n-1$,

$$s(P_n^2, n-1) = s(P_{n-1}^2, n-2) + s(P_{n-2}^2, n-1)$$

$$\Rightarrow x^{n-1} s(P_n^2, n-1) = x^{n-1} s(P_{n-1}^2, n-2) + x^{n-1} s(P_{n-2}^2, n-1)$$

When $i = n$

$$s(P_n^2, n) = s(P_{n-1}^2, n-1) + s(P_{n-2}^2, n)$$

$$\Rightarrow x^n s(P_n^2, n) = x^n s(P_{n-1}^2, n-1) + x^n s(P_{n-2}^2, n)$$

Hence,

$$x^3 s(P_n^2, 3) + x^4 s(P_n^2, 4) + x^5 s(P_n^2, 5) + \ldots\ldots + x^{n-1} s(P_n^2, n-1) + x^n s(P_n^2, n)$$

$$= x^3 s(P_{n-1}^2, 2) + x^3 s(P_{n-2}^2, 3) + x^4 s(P_{n-1}^2, 3) + x^4 s(P_{n-2}^2, 4) + x^5 s(P_{n-1}^2, 4) + x^5 s(P_{n-2}^2, 5) + \ldots\ldots\ldots\ldots$$

$$+ x^{n-1} s(P_{n-1}^2, n-2) + x^{n-1} s(P_{n-2}^2, n-1)$$

$$+ x^n s(P_{n-1}^2, n-1) + x^n s(P_{n-2}^2, n)$$

$$x^3 s(P_n^2, 3) + x^4 s(P_n^2, 4) + x^5 s(P_n^2, 5) + \ldots + x^{n-1} s(P_n^2, n-1) + x^n s(P_n^2, n)$$

$$=[x^3 s(P_{n-1}^2, 2) + x^4 s(P_{n-1}^2, 3) + x^5 s(P_{n-1}^2, 4) + \ldots + x^{n-1} s(P_{n-1}^2, n-2) + x^n s(P_{n-1}^2, n-1)] + [x^3 s(P_{n-2}^2, 3) +$$

$$x^4 s(P_{n-2}{}^2, 4) + x^5 s(P_{n-2}^2, 5) + \ldots + x^{n-1} s(P_{n-2}^2, n-1) + x^n s(P_{n-2}^2, n)]$$

$$x^3 s(P_n^2, 3) + x^4 s(P_n^2, 4) + x^5 s(P_n^2, 5) + \ldots + x^{n-1} s(P_n^2, n-1) + x^n s(P_n^2, n)$$

$=[x^3 s(P_{n-1}^2, 2) + x^4 s(P_{n-1}^2, 3) + x^5 s(P_{n-1}^2, 4) + \ldots\ldots + x^{n-1} s(P_{n-1}^2, n-2) + x^n s(P_{n-1}^2, n-1)]$

$+ [x^3 s(P_{n-2}^2, 3) + x^4 s(P_{n-2}^2, 4) + x^5 s(P_{n-2}^2, 5) + \ldots + x^{n-3} s(P_{n-2}^2, n-3) + x^{n-2} s(P_{n-2}^2, n-2)]$

(Since, $s(P_{n-2}^2, n-1) = s(P_{n-2}^2, n) = 0$)

Hence,

$$\sum_{i=3}^{n} s(P_n^2, i)x^i = x\sum_{i=2}^{n-1} s(P_{n-1}^2, i)x^i + \sum_{i=3}^{n-2} s(P_{n-2}^2, i)x^i$$

ie, $S(P_n^2, x) = x\,S(P_{n-1}^2, x) + S(P_{n-2}^2, x)$.

Hence the theorem. □

Using theorem 2.6, we get $s(P_n^2, i)$ for $2 \le n \le 15$ as shown in the Table 2.1.
Table 2.1: $s(P_n^2, i)$ is the number of Steiner sets of P_n^2 with cardinality i.

n \ i	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	1													
3	0	1												
4	1	0	1											
5	0	2	0	1										
6	1	0	3	0	1									
7	0	3	0	4	0	1								
8	1	0	6	0	5	0	1							
9	0	4	0	10	0	6	0	1						
10	1	0	10	0	15	0	7	0	1					
11	0	5	0	20	0	21	0	8	0	1				
12	1	0	15	0	35	0	28	0	9	0	1			
13	0	6	0	35	0	56	0	36	0	10	0	1		
14	1	0	21	0	70	0	84	0	45	0	11	0	1	
15	0	7	0	56	0	126	0	120	0	55	0	12	0	1

Theorem 2.8

The following properties for the coefficients of $S(P_n^2, x)$ hold:

i) For every $n \ge 2$, $s(P_n^2, n) = 1$,

ii) For every $n \ge 4$, $s(P_n^2, n-2) = n-3$,

iii) For every $n \ge 6$, $s(P_n^2, n-4) = \dfrac{(n-4)(n-5)}{2}$

iv) For every $n \ge 8$, $s(P_n^2, n-6) = \dfrac{(n-5)(n-6)(n-7)}{6}$

v) For every $n \ge 10$, $s(P_n^2, n-8) = \dfrac{(n-6)(n-7)(n-8)(n-9)}{24}$

vi) For every $n \geq 12$, $s\,(P_n^2, n-10) = \frac{(n-7)(n-8)(n-9)(n-10)(n-11)}{120}$

vii) For every $n \geq 1$, $s\,(P_{2n}^2, 2) = 1$

viii) For every $n \geq 1$, $s\,(P_{2n+1}^2, 3) = n$,

ix) For every $n \geq 2$, $s\,(P_{2n}^2, 4) = \frac{n(n-1)}{2}$,

x) For every $n \geq 3$, $s\,(P_{2n-1}^2, 5) = \frac{n(n-1)(n-2)}{6}$,

xi) For every $n \geq 4$, $s\,(P_{2(n-1)}^2, 6) = \frac{n(n-1)(n-2)(n-3)}{24}$

Proof:

i) For any connected graph G with n vertices, the whole set $\{[n]\}$ is a Steiner set. Hence, $s\,(G, n) = 1$.
Therefore, $s\,(P_n^2, n) = 1$, if $n \geq 2$.

ii) We prove by induction on n.
The result is true for $n = 4$, since $s\,(P_4^2, 2) = 1$.
Assume that the result is true for all natural numbers less than n. Now, we prove it for n.
By theorem 2.6,

$$s\,(P_n^2, n-2) = s\,(P_{n-1}^2, n-3) + s\,(P_{n-2}^2, n-2)$$
$$= (n-1) - 3 + 1 \qquad [\text{since } s\,(P_{n-2}^2, n-2) = 1 \text{ by part (i)}]$$
$$= n-3$$

Therefore, the result is true for all n.

iii) We prove by induction on n.
The result is true for $n = 6$, since $s\,(P_6^2, 2) = 1 = \frac{2\times 1}{2}$.
Assume that the result is true for all natural numbers less than n. Now, we prove it for n.
By theorem 2.6,

$$s\,(P_n^2, n-4) = s\,(P_{n-1}^2, n-5) + s\,(P_{n-2}^2, n-4)$$
$$= \frac{(n-5)n-6)}{2} + n - 5 \qquad [\text{ by part (ii)}]$$
$$= \frac{(n-5)(n-6)+2(n-5)}{2}$$
$$= \frac{(n-5)(n-6+2)}{2}$$
$$= \frac{(n-4)(n-5)}{2}$$

Therefore, the result is true for all n.

iv) The result is true for $n = 8$, since $s\,(P_8^2, 2) = 1 = \frac{3\times2\times1}{6}$.

Assume that the result is true for all natural numbers less than n. Now, we prove it for n.

By theorem 2.6,

$$s\,(P_n^2, n-6) = s\,(P_{n-1}^2, n-7) + s\,(P_{n-2}^2, n-6)$$

$$= \frac{(n-6)(n-7)(n-8)}{6} + \frac{(n-6)(n-7)}{2} \quad \text{[by part (iii)]}$$

$$= \frac{(n-6)(n-7)(n-8)+3(n-6)(n-7)}{6}$$

$$= \frac{(n-6)(n-7)[n-8+3]}{6}$$

$$= \frac{(n-5)(n-6)(n-7)}{6}$$

Therefore, the result is true for all n.

v) The result is true for $n = 10$, since $s\,(P_{10}^2, 2) = 1 = \frac{4\times3\times2\times1}{24}$

Assume that the result is true for all natural numbers less than n. Now, we prove it for n.

By theorem 2.6,

$$s\,(P_n^2, n-8) = s\,(P_{n-1}^2, n-9) + s\,(P_{n-2}^2, n-8)$$

$$= \frac{(n-7)(n-8)(n-9)(n-10)}{24} + \frac{(n-7)(n-8)(n-9)}{6} \quad \text{[by part (iv)]}$$

$$= \frac{(n-7)(n-8)(n-9)(n-10)+4(n-7)(n-8)(n-9)}{24}$$

$$= \frac{(n-7)(n-8)(n-9)[n-10+4]}{24}$$

$$= \frac{(n-7)(n-8)(n-9)[n-10+4]}{24}$$

$$= \frac{(n-6)(n-7)(n-8)(n-9)}{24}$$

Therefore, the result is true for all n.

vi) The result is true for $n = 12$, since $s\,(P_{12}^2, 2) = 1 = \frac{5\times4\times3\times2\times1}{120}$

Assume that the result is true for all natural numbers less than n. Now, we prove it for n.

By theorem 2.6,

$$s\,(P_n^2, n-10) = s\,(P_{n-1}^2, n-11) + s\,(P_{n-2}^2, n-10)$$

$$= \frac{(n-8)(n-9)(n-10)(n-11)(n-12)}{120} + \frac{(n-8)(n-9)(n-10)(n-11)}{24} \quad \text{[by part (v)]}$$

$$= \frac{(n-8)(n-9)(n-10)(n-11)(n-12)+5(n-8)(n-9)(n-10)(n-11)}{120}$$

$$= \frac{(n-8)(n-9)(n-10)(n-11)[n-12+5]}{120}$$

$$= \frac{(n-7)(n-8)(n-9)(n-10)(n-11)}{120}$$

Therefore, the result is true for all n.

vii) For every $n \geq 1$, $S(P_{2n}^2, 2)$ has only one Steiner set of cardinality 2 is {1, 2n}.

Therefore, $s(P_{2n}^2, 2) = 1$.

viii) For any square of path P_{2n+1}^2, the Steiner sets of cardinality 3 are {1, 2, 2n + 1}, {1, 4, 2n + 1}, {1, 6, 2n + 1}, …, {1, 2n, 2n + 1}. Therefore, s $(P_{2n+1}^2, 3) = n$.

ix) We prove by induction on n. The result is true for $n = 2$, since $s(P_4^2, 4) = 1 = \frac{2\times 1}{2}$

Assume that the result is true for all natural numbers less than n.

Therefore, $s(P_{2(n-1)}^2, 4) = \frac{(n-1)(n-2)}{2}$ ……………………………… (1)

By part (viii), $s(P_{2n-1}^2, 3) = s(P_{2(n-1)+1}^2, 3) = n - 1$………………… (2)

Now, we prove it for n.

By theorem 2.6,

$$s(P_{2n}^2, 4) = s(P_{2n-1}^2, 3) + s(P_{2n-2}^2, 4)$$

$$= n - 1 + \frac{(n-1)(n-2)}{2}, \qquad \text{[by (1) and (2)]}$$

$$= \frac{2(n-1)+(n-1)(n-2)}{2}$$

$$= \frac{(n-1)(2+n-2)}{2}$$

$$= \frac{(n-1)n}{2}$$

Therefore, the result is true for all n.

x) We prove by induction on n.

The result is true for $n = 3$, since $s(P_5^2, 5) = 1 = \frac{3\times 2\times 1}{6}$

Assume that the result is true for all natural numbers less than n.

Therefore, $s(P_{2(n-1)-1}^2, 5) = \frac{(n-1)\times(n-2)\times(n-3)}{6}$ ………………………… (3)

By part (ix), $s(P_{2n-2}^2, 4) = s(P_{2(n-1)}^2, 4) = \frac{(n-1)\times(n-2)}{2}$ ………………… (4)

Now, we prove it for n.

By theorem 2.6,

$$s(P_{2n-1}^2, 5) = s(P_{2n-2}^2, 4) + s(P_{2n-3}^2, 5)$$

$$= \frac{(n-1)\times(n-2)}{2} + \frac{(n-1)\times(n-2)\times(n-3)}{6}, \qquad \text{[by (3) and (4)].}$$

$$= \frac{(n-1)\times(n-2)\times(3+n-3)}{6}$$

$$= \frac{n\times(n-1)\times(n-2)}{6}$$

Therefore, the result is true for all n.

xi) We prove by induction on n.

The result is true for $n = 4$, since $s\,(P_6^2, 6) = 1 = \frac{4\times3\times2\times1}{24}$ [by part (i)]

Assume that the result is true for all natural numbers less than n.

$$s\,(P_{2(n-1)}^2, 6) = \frac{(n-1)(n-2)(n-3)(n-4)}{24} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

By part (x), $s\,(P_{2n-3}^2, 5) = s\,(P_{2(n-1)-1}^2, 5) = \frac{(n-1)\times(n-2)\times(n-3)}{6}$ ……………… (6)

Now, we prove it for n.

By theorem 2.6,

$$s\,(P_{2(n-1)}^2, 6) = s\,(P_{2(n-1)-1}^2, 5) + s\,(P_{2(n-1)-2}^2, 6)$$

Hence,

$$s\,(P_{2(n-1)}^2, 6) = \frac{(n-1)\times(n-2)\times(n-3)}{6} + \frac{(n-1)(n-2)(n-3)(n-4)}{24} \text{ [by (5) and (6)]}$$

$$= \frac{(n-1)(n-2)(n-3)(4+n-4)}{24}$$

$$= \frac{n(n-1)(n-2)(n-3)}{24}$$

Therefore, the result is true for all n.

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