Numerical Solution of Zeeman Heartbeat Systems: A Nonlinear Model

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Abstract

The mathematical modeling of biological systems has proven to be a valuable tool by allowing experiments which would otherwise be unfeasible in a real situation. In this work we have solved the system of nonlinear differential equations in Zeeman heart model describing the length of a muscle fiber in the heart. In addition, we have used an electrical control variable that triggers the electro-chemical wave leading to the heart contraction using numerical methods ((Scilab) Matlab software). Furthermore, we have employed this methodology to analyze various values of tension, typical length of muscle fiber and positive constant associated with the system. Moreover, the model is able to classify different states, such as diastole which is the relaxed state and systole which is the contracted state. The solution provides a quite complete description of different pathological phenomena and its simplicity can be exploited for further studies on the nonlinear heart beat model.

Keywords: Mathematical Modeling. Nonlinear Equation, Zeeman Heart Beat Model, Numerical Solutions

1. Introduction

The electrocardiogram (ECG) signal is one of the most obvious effects of the human heart operation. The oscillation between systole and diastole states of the heart is reflected in the heart rate. The surface ECG is the recorded potential difference between two electrodes placed on the surface of the skin at pre-defined points. The largest amplitude of a single cycle of the normal ECG is referred to as the R-wave manifesting the depolarization process of the ventricle. The time between successive R-waves has been widely used as a measure of the heart function, and this helps to identify patients at risk for a cardiovascular event or death. Analysis of variations in this time series is known as heart rate variability (HRV) analysis.[2,3]
The development of a dynamical model will provide a useful tool to analyse the effects of various physiological conditions on the profiles of the ECG. The model-generated ECG signals with various characteristics can also be used as signal sources for the assessment of diagnostic ECG signal processing devices. The dynamic response of the cardiovascular control system to physiological changes is reflected in HRV and blood pressure [3]. Recent attention has focused on what HRV signifies in term of cardiovascular health, and HRV is being investigated as a high-risk indicator for possible mortality following myocardial infraction [4]. Some of these fluctuations are relatively well understood and arise from: (i) the interactions between different physiological control mechanisms such as respiratory sinus arrhythmia (RSA) and Mayer waves; (ii) the amount of physical and mental activity; (iii) the circadian rhythm; and (iv) the effects of different sleep stages [2, 5, 6, 7]. So, being able to produce a time series for HRV is an important aspect in generating an artificial ECG signal.

The motivation for using nonlinear methods in modelling HRV is that the source of the mechanism for generating the ECG (i.e. the propagation of electrical activities in myocardium) is nonlinear. In 1972, Zeeman presented an important set of nonlinear dynamical equations for heartbeat modelling [8, 9, 10], based on the Van der Pol-Lienard equation.

Mathematical models [11] are extremely important for understanding biological processes. Now catastrophe theory is one branch of applied mathematics that was developed in order to describe certain biological processes and has been applied by researchers, especially by Christopher Zeeman [8] for a full collection of his works. In Zeeman’s original paper on the heartbeat study [8, 9] he analyzed two sorts of biological excitable systems (nerve and heart). The purpose of this paper was to discuss the Zeeman heart beat model using numerical method for various values of parameters.

2. Mathematical Formulation of the Problem

In order to model the heartbeat we have to be able to incorporate the main properties of the heart in a mathematical model. The properties that are considered as fundamental in the model are: (i) The existence of an equilibrium state (fixed point) corresponding to the diastole (relaxed state of the heart), (ii) There must be a threshold for triggering the process whereby the heart contracts from a diastole to a systole (fully contracted state and another equilibrium state). The model should quickly return to the original equilibrium state after the
systole. Based on the Zeeman model, the simplest system of equations that could describe the heartbeat would be of the form:

$$\varepsilon \frac{dx}{dt} = -(x^3 - Tx + y), \ T > 0 \hspace{1cm} (1)$$

$$\frac{dy}{dt} = x - x_d \hspace{1cm} (2)$$

where the variable \(x\) represents the length of a muscle fiber in the heart and the variable \(y\) represents an electrical control variable that triggers the electro-chemical wave leading to the heart contraction, and \(T > 0\) and \(\varepsilon\) are constants which characterize the heart. \(T\) represents a tension of muscle and is related to blood pressure. The boundary conditions are represented by:

\[ x(0) = 1, \ y(0) = 0 \hspace{1cm} (3) \]

### 3. Numerical Analysis of Heart Beat Model

In Zeeman heart beat model, the length of the muscle fiber in the heart \(x\) and the electrical control variable \(y\) depends upon the parameter \(T\), \(\varepsilon\) and \(x_d\). The parameter \(T\) is related to blood pressure (the higher is blood pressure, the higher is muscle tension). The parameter \(x_d\) is the average muscle length in the diastole.

The system can be linearized about its fixed point and the nature of the linear system’s fixed point can be studied. The diagram of the linear equation can be drawn and studied. For the nonlinear equation it is harder to solve the equations and draw the diagrams. Therefore in this manuscript, the nonlinear equations are solved using Matlab. The heart function has been analyzed for various values of \(T\) and \(\varepsilon\). For the case steady state condition equations (1) and (2) becomes as follows:

$$\left( x^3 - Tx + y \right) = -\varepsilon \frac{dx}{dt} = 0 \hspace{1cm} (4)$$

$$x - x_d = \frac{dy}{dt} = 0 \hspace{1cm} (5)$$

Solving the above equations we obtain the following equations:

$$y = -x_d^3 + Tx_d \hspace{1cm} (6)$$

$$x = x_d \hspace{1cm} (7)$$
This represents the steady state solution of the equations (1) and (2). In the non steady state Zeeman heart beat model, the length of a muscle fiber \((x)\) and the electrical control variable \((y)\) depends upon the parameters \(T\), \(\varepsilon\) and \(x_d\) where as the steady state of Zeeman heart beat model only depends upon the parameter \(T\) and \(x_d\).

Figures 1-12 represents length of a muscle fiber in the heart \((x)\) and an electrical control variable leading to the heart contraction \((y)\) verses time \(t\) for various values of the parameter \(T\), \(\varepsilon\) and \(x_d\). From the figures it is observed that the Zeeman heart beat model attains a steady state value (diastole state or relaxed state) when \(x_d \geq 1\). For example in Fig 1, the steady state value of the length of a muscle fiber and the electro-control variables are \(x_d(-1.024)\) and \(Tx_d - x_d^3(-0.0497)\), respectively. This steady state or diastole state value independent of the variable \(\varepsilon\) (Refer Figs. 1-5). The stable equilibrium point that represents the state of diastole can be determined by changing the value of \(x_d\) and \(T\). From the Figs 6-9 and 11, it is inferred that the heart contracts from a diastole to a systole only when the typical length of a muscle fiber \(x_d < 1\) and for all the values of positive constant \(\varepsilon\) and tension of a muscle fiber \(T\). The number of diastole to a systole state (oscillation of the variable \(x\) and \(y\) with respect to time) increases or heart beat increases when \(T\) and \(\varepsilon\) decreases (refer Figs (7-9)). Fig. 12 displays the phase portrait of the system with \(x_d = 1.024\), \(\varepsilon = 0.1\) and \(T = 1.024\).

The cubic line (dashed curve) represents the steady state of the first equation in (1). When \(x_d = 1.024\), the equilibrium point of the system is represented by: \(\left(x_d, Tx_d - x_d^3\right)\). All trajectories initiated above the cubic line, i.e., \(x^3 - Tx + y > 0\), direct downward toward the equilibrium point along the cubic line. Likewise, all trajectories started below the cubic line which is, \(x^3 - Tx + y < 0\), direct upward toward the equilibrium point along the cubic line. All trajectories end up at the limit cycle around the equilibrium point.
4. Conclusion

The system of nonlinear differential equations in Zeeman heart model has been numerically solved. Two important states such as diastole state (relaxed state) and systole state (contracted state) are discussed with respect to the parameters $x_d$ and $T$. This model will provide a useful tool to measure the variation in the heart function without medical instrument and this helps to identify patients at risk for a cardiovascular event with a minimum cost and time. This method can be easily extended to find the solution of nonlinear equations in third-order nonlinear heartbeat model.

Fig.1 The length of a muscle fiber in the heart ($x$) and the electrical control variable ($y$) versus time $t$ when $e = 0.1, T = 1, x_d = 1.024$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$.

Fig.2 The length of a muscle fiber in the heart ($x$) and the electrical control variable ($y$) versus time $t$ when $e = 1, T = 1, x_d = 1.024$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$. 
Fig. 3 The length of a muscle fiber in the heart (x) and the electrical control variable (y) versus time t when $\varepsilon = 0.001, T = 0.001, x_d = 1$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$.

Fig. 4 The length of a muscle fiber in the heart (x) and the electrical control variable (y) versus time t when $\varepsilon = 0.01, T = 0.01, x_d = 0.1$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$.

Fig. 5 The length of a muscle fiber in the heart (x) and the electrical control variable (y) versus time t when $\varepsilon = 0.01, T = 0.01, x_d = 0.01$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$. 
Fig. 6 The length of a muscle fiber in the heart ($x$) and the electrical control variable ($y$) versus time $t$ when $\varepsilon = 0.01, T = 0.1, x_d = 0$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$.

Fig. 7 The length of a muscle fiber in the heart ($x$) and the electrical control variable ($y$) versus time $t$ when $\varepsilon = 0.01, T = 1, x_d = 0$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$.

Fig. 8 The length of a muscle fiber in the heart ($x$) and the electrical control variable ($y$) versus time $t$ when $\varepsilon = 1, T = 1, x_d = 0$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$. 
Fig. 9 The length of a muscle fiber in the heart \( x \) and the electrical control variable \( y \) versus time \( t \) when \( \varepsilon = 0.1, T = 0, x_d = 1.024 \), for the boundary conditions \( x(0) = 1 \) and \( y(0) = 0 \).

Fig. 10 The length of a muscle fiber in the heart \( x \) and the electrical control variable \( y \) versus time \( t \) when \( \varepsilon = 0.1, T = 0, x_d = 0 \), for the boundary conditions \( x(0) = 1 \) and \( y(0) = 0 \).
Fig. 11 Phase portrait of the first order heart beat system during systole for $\epsilon = 0.1, T = 1, x_d = 1.024$, for the boundary conditions $x(0) = 1$ and $y(0) = 0$.

Appendix A. Numerical Program for the Solution of Nonlinear Eqns. (1) & (2)

```matlab
function graphmain3h
    options=odeset('RelTol',1e-6,'stats','on');
    X0=[1,0];
    tspan=[0,10];
    tic
    [t,X]=ode45(@(t,x)TestFunction(t,x),tspan,X0,options);
    toc
    figure
    hold on
    plot(t,X(:,1));
    figure
    hold on
    plot(t,X(:,2));
    legend('x1','x2')
    ylabel('x')
    xlabel('t')
    return
function [dx_dt]=TestFunction(t,x)
    T=0.1,a=0.1,f=1.024;
    dx_dt(1)=-(1/a)*(x(1)^3-T*x(1)+x(2));
    dx_dt(2)=x(1)-f;
    dx_dt=dx_dt';
    return
```
5 Nomenclatures

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
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<tbody>
<tr>
<td>$x$</td>
<td>Length of a muscle fiber in the heart</td>
</tr>
<tr>
<td>$y$</td>
<td>Electrical control variable that triggers the electro-chemical wave leading to the heart contraction</td>
</tr>
<tr>
<td>$T &gt; 0$</td>
<td>Tension of muscle fiber in the heart</td>
</tr>
<tr>
<td>$x_d$</td>
<td>Scalar quantity representing a typical length of the muscle fiber</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>Small positive constant</td>
</tr>
<tr>
<td>$t$</td>
<td>Time</td>
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References


