n- FOLD IMPLICATIVE V- FILTERS OF LATTICE IMPLICATION ALGEBRAS

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Abstract: We introduce the concept of n- fold implicative V- filters of lattice implication algebras. We discuss the relation among n - fold implicative V - filter and V- filter and some equivalent conditions on n - fold implicative V - filters.

Introduction

In order to research the logical system whose proportional value is given lattice, Y. XU [5] proposed the concept of lattice implication algebras, and discussed their some properties. The concept of vague set introduced by Gau in 1993. Vague sets as an extension of fuzzy sets, the idea of vague sets is that the membership of every element can be divided into two aspects including supporting and opposing. Ya Qin and Yi Liu[3] introduced and discussed the properties of V- filter on lattice implication algebras. Anitha.T, AmarendraBabu. V, [1] introduced the concept of implicative V- filter on lattice implication algebras. At first Y.B.Jun[4] introduced and discussed the concept of n-fold filter. In this paper we introduce the concept of n-fold implicative V- filters and discussed properties of these filters.

2. Preliminaries

In this section we collect important results which were already proved for our use in the next section.

Definition 2.1: [1] Let (L,∨,∧,¬, 0, I) be a complemented lattice with the universal bounds 0, I. → is another binary operation of L. (L,∨,∧,¬, 0, I ) is called a lattice implication algebra, if the following axioms hold, ∀ x, y, z ∈ L,

(I₁) x → (y → z) = y → (x → z);
(I₂) x → x = I;
(I₃) x → y = y' → x';
(I₄) x → y = y → x = I implies x = y;
(I₅) (x → y) → y = (y → x) → x;
(L₁) (x ∨ y) → z = (x → z) ∧ (y → z);
(L₂) (x ∧ y) → z = (x → z) ∨ (y → z).
Definition 2.2: [8] A vague set \(A\) in the universal of discourse \(X\) is characterized by two membership functions given by:

1. A truth membership function \(t_A : X \rightarrow [0,1]\) and
   2. A false membership function \(f_A : X \rightarrow [0,1]\),

Where \(t_A(x)\) is a lower bound of the grade of membership of \(x\) derived from the “evidence for \(x\)”, and \(f_A(x)\) is a lower bound on the negation of \(x\) derived from the “evidence against \(x\)” and \(t_A(x) + f_A(x) \leq 1\). Thus the grade of membership of \(x\) in the vague set \(A\) is bounded by subinterval \([t_A(x), 1 - f_A(x)]\) of \([0, 1]\). The vague set \(A\) is written as

\[A = \{\langle x, [t_A(x), f_A(x)]\rangle / x \in X\}.\]

Where the interval \([t_A(x), 1 - f_A(x)]\) is called the value of \(x\) in the vague set \(A\) and denoted by \(V_A(x)\).

Notation [8]: Let \(I [0, 1]\) denote the family of all closed subintervals of \([0, 1]\). If \(I_1 = [a_1, b_1]\), \(I_2 = [a_2, b_2]\) are two elements of \(I [0, 1]\), we call \(I_1 \geq I_2\) if \(a_1 \geq a_2\) and \(b_1 \geq b_2\). We define the term \(\text{imax}\) to mean the maximum of two intervals as

\[\text{imax} [I_1, I_2] = [\max \{a_1, a_2\}, \max \{b_1, b_2\}]\].

Similarly, we can define the term \(\text{imin}\) of any two intervals.

Definition 2.3: [6] Let \(A\) be a vague set of lattice implication algebra \(L\). \(A\) is said to be a \(v\)-filter of \(L\) if it satisfies the following conditions:

1. \(\forall x \in L, V_A(I) \geq V_A(x)\),
2. \(\forall x, y \in L, V_A(y) \geq \text{imin}\{V_A(xy), V_A(x)\}\).

Definition 2.4: Let \(F\) be a vague set of a lattice implication algebra \(L\). \(F\) is said to be a Implicative \(v\)-filter of \(L\) if it satisfies the following conditions:

1. \(\forall x \in L, V_F(I) \geq V_F(x)\),
2. \(\forall x, y, z \in L, V_F(xz) \geq \text{imin}\{V_F(xy), V_F(x(yz))\}\)

Notation: In this paper we denoted \(p(.....(p(p(pq))))..... = p^nq\) where \(p\) occurs \(n\) times where \(p, q \in L\).

3. \(n\)-fold implicative \(V\)-filters

Definition 3.1: Let \(H\) be a vague set of a lattice implication algebra \(L\). \(H\) is said to be a \(n\)-fold implicative \(V\)-filter of \(L\) if it satisfies the following conditions:

1. \(\forall p \in L, V_H(I) \geq V_H(p)\),
2. \(\forall p, q, r \in L, V_H(p^nr) \geq \text{imin}\{V_H(p^n(qr)), V_H(p^nq)\}\) \(.....3.1\)

Note: If \(n = 1\) in the equation 3.1, we get the vague 1-fold implicative filter. Actually that is a implicative \(v\)-filter of \(L\).

Theorem 3.2: Vague set \(H\) of \(L\) is a \(n\)-fold implicative \(V\)-filter of \(L\), if and only if, for any \(p, q, r \in L\)

1. \(t_H(I) \geq t_H(x)\) and \(L - f_H(I) \geq 1 - f_H(x)\);
2. \(t_H(p^nr) \geq \text{imin}\{t_H(p^n(qr)), t_H(p^nq)\}\)

and
1 - \( f_H(p^r) \geq \min\{ 1 - f_H(p^r (qr)), 1 - f_H(p^r q) \} \)

**Example 3.3:** Let \( L = \{0, a, b, c, d, I\} \) be a set with Cayley table as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>A</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>I</td>
<td>I</td>
<td>I</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>A</td>
<td>C</td>
<td>I</td>
<td>b</td>
<td>c</td>
<td>b</td>
<td>I</td>
</tr>
<tr>
<td>B</td>
<td>D</td>
<td>A</td>
<td>I</td>
<td>b</td>
<td>a</td>
<td>I</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>A</td>
<td>I</td>
<td>a</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>D</td>
<td>B</td>
<td>I</td>
<td>l</td>
<td>b</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>I</td>
<td>0</td>
<td>A</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>I</td>
</tr>
</tbody>
</table>

Define \( \rightarrow, \lor \) and \( \land \) operations on \( L \) as follows:

\[
x' = x \rightarrow 0, \quad x \lor y = (x \rightarrow y) \rightarrow y,
\]

\[
x \land y = ((x' \rightarrow y') \rightarrow y')' \quad \text{for all } x, y \in L.
\]

Then \((L,\lor, \land, \rightarrow, \lor, 0, I)\) is a lattice implication algebra [5]. Let \( H = (t_H, f_H) \) be a vague set on \( L \) defined by

\[
t_H(p) = 0.72 \quad \text{if } p = b, c, I
\]

\[
= 0.53 \quad \text{if } p = 0, a, d
\]

\[
f_H(p) = 0.21 \quad \text{if } p = b, c, I
\]

\[
= 0.32 \quad \text{if } p = 0, a, d
\]

Clearly the vague set \( H \) is a \( n \)-fold implicative \( \lor \)-filter of \( L \).

**Theorem 3.4:** Every \( n \)-fold implicative \( \lor \)-filter of \( L \) is a \( \lor \)-filter of \( L \).

**Proof:** Let \( H \) be a \( n \)-fold implicative \( \lor \)-filter of \( L \).

Then \( V_H(p^r) \geq \min\{ V_H(p^r (qr)), V_H(p^r q) \} \) for all \( p, q, r \in L \).

Taking \( p = I \) in the equation, we get

\[
V_H(r) \geq \min\{ V_H(qr), V_H(q) \}
\]

Hence \( H \) is a \( \lor \)-filter of \( L \).

**Remark 3.5:** The converse of the theorem 3.3 may not be true. For example \( G = (t_G, f_G) \) is a vague set on \( L \) in the example 3.3 as follows:

\[
t_G(p) = 0.31 \quad \text{if } p = I
\]

\[
= 0.28 \quad \text{if } p = 0, a, b, c, d
\]

\[
f_G(p) = 0.43 \quad \text{if } p = I
\]

\[
= 0.51 \quad \text{if } p = 0, a, d, c, d
\]

Clearly the vague set \( G = (t_G, f_G) \) is a \( \lor \)-filter of \( L \). But it is not a \( 1 \)-fold implicative \( \lor \)-filter of
L because \(0.28 = \text{t}_{11}(d^1c) \geq \text{imin}\{\text{t}_{11}(d^1(bc)), \text{t}_{11}(d^1b))\}
\[
= 0.31.
\]

We give a condition for a \(V\)-filter to be \(n\)-fold implicative \(V\)-filter.

**Theorem 3.6:** A \(V\)-filter \(H\) of \(L\) is a \(n\)-fold implicative \(V\)-filter of \(L\) if and only if

\[
\text{V}_H(p^nq) \geq \text{V}_H(p^{n+1}q) \text{ for all } p, q \in L.
\]

**Proof:** Suppose that \(H\) is a \(n\)-fold implicative \(V\)-filter of \(L\). Then

\[
\text{V}_H(p^nq) \geq \text{imin}\{\text{V}_H(p^n(qr)), \text{V}_H(p^nq)\} \text{ for all } p, q, r \in L.
\]

Taking \(q = p\) in the equation, we get

\[
\text{V}_H(p^nr) \geq \text{imin}\{\text{V}_H(p^n(pr)), \text{V}_H(p^nq)\}
\]

\[
= \text{imin}\{\text{V}_H(p^{n+1}r), \text{V}_H(p^{n+1})\}
\]

\[
= \text{imin}\{\text{V}_H(p^{n+1}r), \text{V}_H(I)\}
\]

\[
= \text{V}_H(p^{n+1}r)
\]

Conversely suppose that a \(V\)-filter satisfies the inequality \(\text{V}_H(p^nq) \geq \text{V}_H(p^{n+1}q)\) for all \(p, q \in L\).

Then \(\text{V}_H(p^nq) \geq \text{V}_H(p^{n+1}q)\)

\[
\Rightarrow \text{V}_H(p^nq) \geq \text{V}_H(p^nq)
\]

\[
\Rightarrow \text{V}_H(q^n) \geq \text{V}_H(p^nq)
\]

\[
\Rightarrow \text{V}_H(p^nq) \geq \text{V}_H(p^nq)
\]

Then clearly \(\text{V}_H(q^n) \geq \text{V}_H(p^nq) \) ………… 3.2

\(H\) is a \(V\)-filter of \(L\) and from the inequality 3.2, it follows

\[
\text{V}_H(p^nq) \geq \text{imin}\{\text{V}_H(p^n(qr)), \text{V}_H(q^n)\}
\]

\[
= \text{imin}\{\text{V}_H(p^n(qr)), \text{V}_H(p^nq)\}
\]

Hence \(H\) is a \(n\)-fold implicative \(V\)-filter of \(L\).

**Theorem 3.7:** A vague set \(H\) of \(L\) is a \(n\)-fold implicative \(V\)-filter of \(L\) if and only if

\[
\text{V}_H(q^nr) \geq \text{imin}\{\text{V}_H(p(q^{n+1}r)), \text{V}_H(p)\} \text{ for all } p, q, r \in L.
\]

**Proof:** Suppose that \(H\) is a \(n\)-fold implicative \(V\)-filter of \(L\). So

\[
\text{V}_H(p^nq) \geq \text{V}_H(p^{n+1}q) \text{ for all } p, q \in L \) …………3.3
\]

Since \(H\) is a \(V\)-filter of \(L\), then we have \(\text{V}_H(q^{n+1}r) \geq \text{imin}\{\text{V}_H(p(q^{n+1}r)), \text{V}_H(p)\}\) …3.4

From the inequalities 3.3 and 3.4, we get \(\text{V}_H(p^nq) \geq \text{imin}\{\text{V}_H(p(q^{n+1}r)), \text{V}_H(p)\}\).

Conversely suppose that \(H\) is a vague set of \(L\) such that \(\text{V}_H(I) \geq \text{V}_H(p)\) and satisfies the inequality

\[
\text{V}_H(q^nr) \geq \text{imin}\{\text{V}_H(p(q^{n+1}r)), \text{V}_H(p)\} \) …3.5
Taking \( q = I \) in 3.5, we get
\[
V_H(r) \geq \text{imin}\{V_H(p \ r), \ V_H(p)\},
\]
so \( H \) is a \( V \)-filter of \( L \).

Taking \( p = I \) in 3.5, we get
\[
V_H(q^n r) \geq \text{imin}\{V_H(q^{n+1} r), \ V_H(I)\}
\geq V_H(q^{n+1} r)
\]
By theorem 3.6 \( H \) is a \( n \)-fold implicative \( V \)-filter of \( L \).


